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DISCRETE TORSION FOR THE SUPERSINGULAR ORBIFOLD SIGMA GENUS

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Abstract. The first purpose of this paper is to examine the relationship between equivariant elliptic genera and orbifold elliptic genera. We apply the character theory of [HKR00] to the Borel-equivariant genus associated to the sigma orientation of [AHS01] to define an orbifold genus for certain total quotient orbifolds and supersingular elliptic curves. We show that our orbifold genus is given by the same sort of formula as the orbifold "two-variable" genus of [DMVV97] and [BL02]. In the case of a finite cyclic orbifold group, we use the characteristic series for the two-variable genus in the formulae of [And03] to define an analytic equivariant genus in Grojnowski's equivariant elliptic cohomology, and we show that this gives *precisely* the orbifold two-variable genus. The second purpose of this paper is to study the effect of varying the $BU\langle 6\rangle$-structure in the Borel-equivariant sigma orientation. We show that varying the $BU\langle 6\rangle$ structure by a class in $H^3(BG;\mathbb{Z})$, where G is the orbifold group, produces discrete torsion in the sense of [Vaf85]. This result was first obtained by Sharpe [Sha], for a different orbifold genus and using different methods.

1. Introduction

Let E be an even periodic, homotopy commutative ring spectrum, let C be an elliptic curve over $S_E = \operatorname{spec}\pi_0 E$, and let t be an isomorphism of formal groups

$$t : \widehat{C} \cong \operatorname{spf} E^0(\mathbb{C}P^\infty),$$

so that $\mathbf{C} = (E, C, t)$ is an elliptic spectrum in the sense of [Hop95, AHS01]. In [AHS01], Hopkins, Strickland, and the first author construct a map of homotopy commutative ring spectra

$$\sigma(\mathbf{C}) : MU\langle 6\rangle \to E$$

called the *sigma orientation*; it is conjectured in [Hop95] that this map is the restriction to $MU\langle 6\rangle$ of a similar map $MO\langle 8\rangle \to E$.

The sigma orientation is natural in the elliptic spectrum, and, if $K_{\mathrm{Tate}} = (K[\![q]\!], \mathrm{Tate}, t)$ is the elliptic spectrum associated to the Tate elliptic curve, then the map of homotopy rings

$$\pi_* MU\langle 6\rangle \to \pi_* K_{\mathrm{Tate}} \tag{1.1}$$

is the restriction from $\pi_* MSpin$ of the *Witten genus*. Explicitly, let M be a Riemannian spin manifold, and let D be its Dirac operator. Let T denote the tangent bundle of M. If V is a (real or complex) vector bundle over M, let $V^{\mathbb{C}}$ be the complex vector bundle

$$V^{\mathbb{C}} = V \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

If V is a complex vector bundle, let rV be the reduced bundle

$$rV = V - \underline{\operatorname{rank} V},$$

and let

$$S_t V = \sum_{k\geq 0} t^k S^k V$$

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be the indicated formal power series in the symmetric powers of V. The operation $V \mapsto S_t V$ extends to an exponential operation

$$K(X) \to K(X)[\![t]\!]$$

because of the formula

$$S_t(V \oplus W) = (S_t V)(S_t W).$$

The *Witten genus* of M is given by the formula

$$w(M) = \operatorname{ind}\left(D \otimes \bigotimes_{k \geq 1} S_{q^k}(rT^{\mathbb{C}}) \right) \in \mathbb{Z}[\![q]\!], \tag{1.2}$$

and the diagram

$$\begin{tikzcd} \pi_* MU\langle 6\rangle \arrow[r] \arrow[dr, "\pi_*\sigma(K_{\mathrm{Tate}})"'] & \pi_* MSpin \arrow[d, "w"] \\ & \mathbb{Z}[\![q]\!] \end{tikzcd}$$

commutes [AHS01].

The Witten genus first arose in [Wit87], where Witten showed that various elliptic genera of a manifold M are essentially one-loop amplitudes of quantum field theories of closed strings moving in M. Locally on M the quantum field theory associated to $w(M)$ is a conformal field theory, and the obstruction to assembling a conformal field theory globally on M is $c_2 M$. [1] This gives a physical proof that, if $c_2 M = 0$, then $w(M)$ is the q-expansion of a modular form.

Suppose that M is an SU-manifold. The formula (1.2) shows that the Witten genus is an invariant of the spin structure of M. On the other hand the sigma orientation depends on a choice of $BU\langle 6\rangle$ structure, that is, a lift in the diagram

$$\begin{tikzcd} & BU\langle 6\rangle \arrow[d] & \\ M \arrow[r] \arrow[ur, dashed] & BSU \arrow[r, "c_2"] & K(\mathbb{Z}, 4). \end{tikzcd}$$

It is an interesting problem to understand how the orientation depends on this choice. The fibration sequence

$$K(\mathbb{Z}, 3) \xrightarrow{\iota} BU\langle 6\rangle \to BSU \xrightarrow{c_2} K(\mathbb{Z}, 4)$$

shows that a lift exists precisely when $c_2(M) = 0$, and that the set of lifts is a quotient of $H^3(M; \mathbb{Z})$.

The dependence on the choice of lift appears to have an explanation in string theory. The action for the QFT described by the Witten genus is a function on the space of maps

$$X : \Sigma \to M$$

of 2-dimensional surfaces Σ to M. If the theory is anomaly-free, that is, if $c_2 M = 0$, then one is free to add to the action a term of the form

$$\int_\Sigma X^* B,$$

where $B \in \Omega^2 M$ is a differential 2-form on M (called the "B-field"), provided that

$$H = dB$$

is an *integral* three-form. It seems clear that the physics of the B-field should account for the variation in the sigma orientation from at least torsion classes in $H^3(M, \mathbb{Z})$.

In this paper we provide some evidence for this assertion. We show that varying the $BU\langle 6\rangle$ structure in the *orbifold* sigma genus of a supersingular elliptic curve produces the phenomenon known as *discrete torsion*, so named by Vafa in [Vaf85]. Eric Sharpe has shown that discrete torsion arises from the action

[1] There are various ways to understand this obstruction ([Wit87, BM94, GMS00, And03]).

of the orbifold group on the B-field. Putting Sharpe's results together with ours suggests that, indeed, the B-field is the physical reflection of the choice of $BU\langle 6\rangle$ structure.

More precisely, suppose that M is a complex manifold with an action by a group G, and suppose that V is a complex G-vector bundle over M. Let T denote the tangent bundle of M. If X is a space, let X_G denote the Borel construction $EG \times_G X$. If

$$\begin{aligned} c_1(T_G) &= c_1(V_G) \\ c_2(T_G) &= c_2(V_G), \end{aligned}$$

then there is a lift in the diagram

$$\begin{tikzcd} & BU\langle 6\rangle \arrow[d] \\ M_G \arrow[ur, dashed, "\ell"] \arrow[r, "V_G - T_G"'] & BSU, \end{tikzcd} \tag{1.3}$$

and a choice of lift gives a Thom class

$$U(M, \ell, \mathbf{C})_G \in E^0(V_G - T_G).$$

The relative zero section together with the Pontrjagin-Thom construction provide a map

$$\tau(V)_G : E^0(V_G - T_G) \to E^0(-T_G) \to E^0(BG),$$

and

$$\tau(V)_G(U(M, \ell, \mathbf{C})_G) \in E^0(BG)$$

is the (Borel) equivariant sigma genus of M twisted by V (see §4).

To get from it an "orbifold" genus taking its values in E^0, we use the character theory of Hopkins, Kuhn, and Ravenel ([HKR00]; see also §5). It associates to a pair (g, h) of commuting elements of G a ring homomorphism

$$\Xi_{g,h} : E^0(BG) \to D,$$

where D is a complete local E-algebra which depends on the formal group of the spectrum E. It turns out that the quantity

$$\sigma_{\text{orb}}(M, \ell, \mathbf{C})_G = \sum_{gh=hg} \Xi_{g,h}\tau(V)_G(U(M, \ell, \mathbf{C})_G) \tag{1.4}$$

takes its values in E^0; we call it the *orbifold sigma genus* of M twisted by V (see §6).

There is already an extensive literature on the subject of "orbifold elliptic genera", particularly the "two-variable" elliptic genus of [Kri90, EOTY89]; see for example [DMVV97, BL02]. In §6, we show that the formula (1.4) is formally analogous to the formula for the orbifold two-variable genus. It is difficult to make a more precise comparison between our two situations, because we work with the Borel-equivariant elliptic cohomology associated to a supersingular elliptic curve, which is a highly completed situation.

In order to locate the orbifold two-variable genus more precisely in the setting of equivariant elliptic cohomology, we consider in §7 the case of a finite cyclic group $G = \mathbb{T}[n] \subset \mathbb{T}$. We use the principle suggested by Shapiro's Lemma to define

$$E_G(X) \overset{\text{def}}{=} E_{\mathbb{T}}(\mathbb{T} \times_G X),$$

where $E_{\mathbb{T}}$ is the uncompleted analytic equivariant elliptic cohomology of Grojnowski. We adapt the formulae in [And03], which descends from [Ros01, AB02], to write down an euler class in $E_G(X)$. The associated genus $\mathrm{Ell}^{an}(M, G)$ takes its value in $\Gamma(E_G(*)) \cong \Gamma(\mathcal{O}_{C[n]}) \cong \mathbb{C}^{G\times G}$, and we prove the following.

Theorem 1.5. *Summing the analytic equivariant two-variable genus over the torsion points of the elliptic curve gives the orbifold two-variable genus: more precisely, we have*

$$\mathrm{Ell}_{orb}(X, G) = \frac{1}{|G|} \sum_{gh=hg} \mathrm{Ell}^{an}(M, G, g, h).$$

We were pleased to be able to confirm that orbifold elliptic genera are so simply obtained from equivariant elliptic genera. It would be interesting to use this observation to investigate more subtle properties of orbifold genera, such as, for example, the "McKay correspondence" of Borisov and Libgober.

The rest of the paper is devoted to the study of the dependence of the orbifold sigma genus on the choice ℓ of $BU\langle 6\rangle$ structure in (1.3). Suppose that we have chosen an element $u \in H^3(BG;\mathbb{Z})$, represented as a map

$$u : BG \to K(\mathbb{Z},3).$$

If $\pi : M_G \to BG$ denotes the projection in the Borel construction, then we obtain an element

$$\pi^* u = u\pi \in H^3(M_G;\mathbb{Z}),$$

and $\ell + \iota u\pi$ is another $BU\langle 6\rangle$-structure on $V_G - T_G$.

In §8, we use the character theory and the sigma orientation to associate to u an alternating bilinear map

$$\delta = \delta(u,\mathbf{C}) : G_2 \to D^\times,$$

where G_2 denotes the set of pairs of commuting elements of G of p-power order. In §9 we obtain the

Theorem 1.6. *The orbifold sigma genus associated to the $BU\langle 6\rangle$ structure $\ell+\iota u\pi$ is related to the equivariant sigma genus associated to ℓ by the formula*

$$\sigma_{\mathrm{orb}}(M,\ell+\iota u\pi,\mathbf{C})_G = \sum_{gh=hg} \delta(g,h)\Xi_{g,h}\tau(V)_G(U(M,\ell,\mathbf{C})_G).$$

In [Vaf85], Vafa observed that if

$$\phi = \sum_{gh=hg} \phi_{g,h}$$

is an orbifold elliptic genus associated to a theory of strings on M, and if $c = c(g,h)$ is a 2-cocycle with values in $U(1)$, then

$$\sum_{gh=hg} c(g,h)\phi_{g,h} \tag{1.7}$$

is again modular; he called this phenomenon "discrete torsion". Eric Sharpe [Sha] showed that the genus (1.7) arises from adding a B-field

$$B \in \Omega^2(M_G)$$

such that

$$dB = [c] \in H^3(M_G;\mathbb{Z}),$$

where $[c]$ is the cohomology class in M_G obtained from c by pulling back along $M_G \to BG$.

Our result shows that varying the $BU\langle 6\rangle$-structure of M_G by an element $u \in H^3(BG)$ has a similar effect on the orbifold sigma genus. When G is an abelian of order dividing $n = p^s$, the map δ may be viewed as a two-cocycle on G with values in $D^\times[n] \cong \mathbb{Z}/n$, and as such it represents a cohomology class in $H^2(BG;\mathbb{Z}/n) \cong H^3(BG)$. It is not quite the cohomology class u: instead, as we shall see in §10, if c is a 2-cocycle representing $u \in H^3(BG;\mathbb{Z}) \cong H^2(BG;\mathbb{Z}/n)$, then

$$\delta(g,h) = c(g,h) - c(h,g).$$

2. The sigma orientation and the sigma genus

In this section we recall some results from [AHS01].

Definition 2.1. An *elliptic spectrum* consists of

(1) an even, periodic, homotopy commutative ring spectrum E with formal group $P_E = \operatorname{spf} E^0\mathbb{C}P^\infty$ over $\pi_0 E$;
(2) a generalized elliptic curve C over $\pi_0 E$;
(3) an isomorphism $t : P_E \to \widehat{C}$ of P_E with the formal completion of C.

A *map* (f,s) of elliptic spectra $\mathbf{E_1} = (E_1, C_1, t_1) \to \mathbf{E_2} = (E_2, C_2, t_2)$ consists of a map $f : E_1 \to E_2$ of multiplicative cohomology theories, together with an isomorphism of elliptic curves

$$C_2 \xrightarrow{s} (\pi_0 f)_* C_1,$$

extending the induced isomorphism of formal groups.

Theorem 2.2. *An elliptic spectrum* $\mathbf{C} = (E, C, t)$ *determines a map*

$$\sigma(\mathbf{C}) : MU\langle 6\rangle \to E$$

of homotopy-commutative ring spectra. The association $\mathbf{C} \mapsto \sigma(\mathbf{C})$ *is modular, in the sense that if*

$$(f,s) : \mathbf{C_1} \to \mathbf{C_2}$$

is a map of elliptic spectra, then the diagram

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r, "\sigma(\mathbf{C_1})"] \arrow[dr, "\sigma(\mathbf{C_2})"'] & E_1 \arrow[d, "f"] \\ & E_2 \end{tikzcd}$$

commutes up to homotopy. If $K_{\mathrm{Tate}} = (K[\![q]\!], \mathrm{Tate}, t)$ *is the elliptic spectrum associated to the Tate curve, then the diagram*

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r] \arrow[dr, "\sigma(K_{\mathrm{Tate}})"'] & MSpin \arrow[d, "w"] \\ & K[\![q]\!] \end{tikzcd}$$

commutes, where w *is the orientation associated to the Witten genus.*

3. The sigma genus

Definition 3.1. Let W be a virtual complex vector bundle on a space M. A $BU\langle 6\rangle$*-structure* on W is a map

$$\ell : M \to BU\langle 6\rangle$$

such that the composition

$$M \xrightarrow{\ell} BU\langle 6\rangle \to BU$$

classifies rW.

Now let M be a connected compact closed manifold with complex tangent bundle T, and let V be another complex vector bundle on M. Let

$$d = 2\,\mathrm{rank}_{\mathbb{C}}\, T - V$$

Let

$$\tau(V) : S^0 \xrightarrow{P-T} M^{-T} \xrightarrow{\zeta} M^{V-T}$$

be the composition of the Pontrjagin-Thom map with the relative zero section.

If

$$\ell : M \to BU\langle 6\rangle$$

is a $BU\langle 6\rangle$-structure on $V - T$, and if $\mathbf{C} = (E, C, t)$ is an elliptic spectrum, let $U(M, \ell, \mathbf{C}) \in E^{-d}(M^{V-T})$ be the class given by the map

$$U(M, \ell, \mathbf{C}) : \Sigma^d M^{V-T} \xrightarrow{\ell} MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

Definition 3.2. The *sigma genus* of ℓ in $\mathbf{C}$ is the element

$$\sigma(M, \ell, \mathbf{C}) \overset{\mathrm{def}}{=} \tau(V)^*(U(M, \ell, \mathbf{C})) \in E^{-d}(S^0) = \pi_d E.$$

Example 3.3. Suppose that $c_1 T = 0 = c_2 T$, so that T itself admits a $BU\langle 6\rangle$-structure, say $\ell : M \to BU\langle 6\rangle$, and $d = 2\dim M$. Then we have a Thom isomorphism

$$\begin{aligned} E^0(M) &\cong E^{-d}(M^{-T}) \\ 1 &\mapsto U(M,\ell,\mathbf{C}). \end{aligned}$$

and the usual Umkehr map $\pi_!^M$ associated to the projection

$$\pi^M : M \to *$$

is the composition

$$\pi_!^M : E^0(M) \xrightarrow{\cong} E^{-d}(M^{-T}) \xrightarrow{P-T} E^{-d}(S^0) \cong \pi_d E.$$

Thus

$$\sigma(M,\ell,\mathbf{C}) = \pi_!^M(1) = \pi_d(\sigma(\mathbf{C}))([M]) \in \pi_d E$$

is just the genus of M with $BU\langle 6\rangle$-structure ℓ, associated to the sigma orientation

$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

4. The Borel-equivariant sigma genus

Now suppose that G is a compact Lie group, and, if X is a space, let X_G denote the Borel construction

$$X_G \overset{\text{def}}{=} EG \times_G X.$$

Suppose that G acts on the compact connected manifold M, that V is an equivariant complex vector bundle, and that

$$\ell : M_G \to BU\langle 6\rangle$$

is a $BU\langle 6\rangle$-structure on the bundle $V_G - T_G$. Since T_G is the bundle of tangents along the fiber of

$$M_G \to BG,$$

we have a Pontrjagin-Thom map

$$BG_+ \xrightarrow{P-T} (M_G)^{-T_G},$$

and so a map

$$\tau(V)_G : BG_+ \xrightarrow{P-T} (M_G)^{-T_G} \xrightarrow{\zeta} (M_G)^{V_G - T_G}.$$

Let $U(M,\ell,\mathbf{C})_G \in E^{-d}(M_G^{V_G - T_G})$ be given by the map

$$U(M,\ell,\mathbf{C})_G : \Sigma^d (M_G)^{V_G - T_G} \xrightarrow{\ell} MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

Definition 4.1. The *(Borel) equivariant sigma genus* of ℓ in $\mathbf{C}$ is the element

$$\sigma(M,\ell,\mathbf{C})_G \overset{\text{def}}{=} \tau(V)_G(U(M,\ell,\mathbf{C})_G) \in E^{-d}(BG).$$

5. Character theory

The equivariant sigma genus described in §4 is not so familiar, because $E^*(BG)$ is not. In this section we review the character theory of [HKR00], which gives a sensible way to understand $E^*(BG)$. In the next section, we apply the character theory to produce the orbifold sigma genus from the equivariant sigma genus; as we shall see, it is given by the same sort of formula as those for "orbifold elliptic genera" in for example [DMVV97, BL02]

We suppose that E is an even periodic ring spectrum, and that $\pi_0 E$ is a complete local ring of residue characteristic $p > 0$. We write P for $\mathbb{C}P^\infty$, so $P_E = \operatorname{spf} E^0 P$ is the formal group of E. We assume that P_E has finite height h.

Let $\Lambda_\infty = (\mathbb{Z}_p)^h$, and for $n \geq 1$, let $\Lambda_n = \Lambda_\infty / p^n \Lambda_\infty$. If A is an abelian group, let $A^* \overset{\text{def}}{=} \hom(A, \mathbb{C}^\times)$ denote its group of complex characters, so for example $\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h \cong (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^h$.

Each $\lambda \in \Lambda_n^*$ defines a map

$$B\Lambda_n \xrightarrow{B\lambda} P.$$

Choose a coordinate $x \in E^0 P$. For each $\lambda \in \Lambda_n^*$, let

$$x(\lambda) = (B\lambda)^* x \in E^0 B\Lambda_n.$$

Let $S \subset E^0 B\Lambda_n$ be the multiplicative subset generated by $\{x(\lambda) | \lambda \neq 0\}$. Let $L_n = S^{-1} E^0 B\Lambda_n$, and let D_n be the image of $E^0 B\Lambda_n$ in L_n. In other words, D_n is the quotient of $E^0 B\Lambda_n$ by the ideal generated by annihilators of euler classes of non-zero characters of Λ_n. It is clear that L_n and D_n are independent of the choice of coordinate x.

Now suppose that G is a finite group. Let

$$\alpha : \Lambda_n \to G$$

be a homomorphism: specifying such α is equivalent to specifying an h-tuple of commuting elements of G of order dividing p^n.

Definition 5.1. The *character map associated to* α is the ring homomorphism

$$\Xi_\alpha : E^0 BG \xrightarrow{E^0 B\alpha} E^0 B\Lambda_n \to D_n.$$

One may check directly from the definition that the map $\Lambda_{n+1} \twoheadrightarrow \Lambda_n$ induces maps

$$\begin{aligned} D_n &\to D_{n+1} \\ L_n &\to L_{n+1}. \end{aligned}$$

Let

$$\begin{aligned} D &= \operatorname*{colim}_n D_n \\ L &= \operatorname*{colim}_n L_n. \end{aligned} \tag{5.2}$$

Since G is finite, any homomorphism

$$\alpha : \Lambda \to G$$

factors as

$$\alpha : \Lambda \to \Lambda_n \xrightarrow{\alpha_n} G$$

for sufficiently large n, and we may unambiguously attach a character homomorphism

$$\Xi_\alpha : E^0 BG \to D$$

such that, for sufficiently large n, the diagram

$$\begin{tikzcd} E^0 BG \arrow[r, "\Xi_{\alpha_n}"] \arrow[dr, "\Xi_\alpha"'] & D_n \arrow[d] \\ & D \end{tikzcd}$$

commutes.

A great deal is known about the ring D_n, because it turns out [AHS03] that spf D_n is the scheme of *level* Λ_n^**-structures* on the P_E. For example, it is easy to check that the action of $\operatorname{Aut}(\Lambda_n)$ on $E^0 B\Lambda_n$ induces an action of $\operatorname{Aut}(\Lambda_n)$ on D_n. Using the description of D_n in terms of level structures, one may prove the following.

Proposition 5.3. *The ring D_n is finite and faithfully flat over E. If P_E is the universal deformation of a formal group of height h (i.e. if E is a Morava E-theory), then D_n is a complete Noetherian local domain, and in that case, and in general if p is regular in $\pi_0 E$, then $L_n = \frac{1}{p} D_n$. The structural map*

$$E^0 \to D_n$$

identifies E^0 with the $\operatorname{Aut}(\Lambda_n)$*-invariants in D_n.*

Proof. [Dri74] or [Str97]. □

If $w \in \operatorname{Aut}(\Lambda_n)$ and $\alpha : \Lambda_n \to G$, then we have two homomorphisms from $E^0(BG)$ to D_n, namely $w\Xi_\alpha$ and $\Xi_{\alpha w}$.

Lemma 5.4. *The diagram*

$$\begin{tikzcd} E^0BG \arrow[r, "\Xi_\alpha"] \arrow[dr, "\Xi_{\alpha w}"'] & D_n \arrow[d, "w"] \\ & D_n \end{tikzcd}$$

commutes. □

Corollary 5.5. *The expression*

$$\sum_{\alpha:\Lambda_n \to G} \Xi_\alpha$$

defines an additive map

$$E^0BG \to \pi_0 E.$$

□

In the case that the height of P_E is two, the sum is over all pairs of commuting elements of G of p-power order. If G is a p-group, then we write

$$\sum_{gh=hg} \Xi_{g,h} : E^0BG \to \pi_0 E$$

for the map in the Corollary.

6. The orbifold sigma genus

There has been much study of the orbifold version of the two-variable elliptic genus of [EOTY89]; see for example [DMVV97, BL02]. In this section we introduce an orbifold version of the sigma genus, in the case of a supersingular elliptic curve. Our definition is intentionally as simple as possible: we consider only total quotient orbifolds, and then extract the orbifold sigma genus from the Borel genus using the map of Corollary 5.5.

Explicitly, suppose that G is a finite group acting on a manifold M with complex tangent bundle T, that V is an equivariant complex vector bundle, and that

$$\ell : M_G \to BU\langle 6\rangle$$

is a $BU\langle 6\rangle$-structure on the bundle $V_G - T_G$.

Let C be the universal deformation of a supersingular elliptic curve over a perfect field of characteristic $p > 0$, and let $\mathbf{C} = (E, C, t)$ be the associated elliptic spectrum.

Definition 6.1. The *orbifold sigma genus* of ℓ in $\mathbf{C}$ is the element

$$\sigma_{\mathrm{orb}}(M, \ell, \mathbf{C})_G \overset{\mathrm{def}}{=} \sum_{gh=hg} \Xi_{g,h}\sigma(M, \ell, \mathbf{C})_G \in \pi_{-d}E. \tag{6.2}$$

The rest of this section is devoted to showing that the formula (6.2) is formally analogous to the formula for the orbifold two-variable genus. In section 7, we show that, in the case of a finite cyclic group, the orbifold two-variable genus is precisely the genus in Grojnowski's circle-equivariant elliptic cohomology obtained from the characteristic series defining the two-variable genus by following the construction of [Ros01, AB02, And03]. These sections are logically independent of the discussion of discrete torsion and the proof of Theorem 1.6, and readers interested primarily in that formula may prefer to skip to section 8.

Our comparison in this section is based on the analogue of the formula (6.2) in the case of a genus given by a complex orientation

$$t : MU \to E,$$

so that E has Thom classes and Umkehr maps for complex vector bundles.

If M is a compact manifold of real dimension d, then we write π^M for the projection

$$M \to *,$$

and π_t^M for the Umkehr map

$$\pi_t^M : E^*(M) \to \pi_{d-*}E.$$

This Umkehr map is often denoted $\pi_!^M$; our notation emphasizes the dependence on the orientation t. In any case, the genus associated to t is the map

$$\Phi^t : \pi_* MU \to \pi_* E$$

given by the formula

$$\Phi^t(M) = \pi_t^M(1).$$

If a compact Lie group G acts on M, then we write $\pi^{M,G}$ for the projection

$$\pi^{M,G} : M_G \to BG,$$

and $\pi_t^{M,G}$ for the associated Umkehr map

$$\pi_t^{M,G} : E^0(M_G) \to E^{-d}(BG).$$

If G is a finite p-group, then the analogue of our formula (6.2) is the quantity

$$\Phi^t_{\mathrm{orb}}(M) = \sum_{gh=hg} \Xi_{g,h} \pi_t^{M,G}(1).$$

If g and h are commuting elements of G, let

$$M^{(g,h)} \overset{\mathrm{def}}{=} M^g \cap M^h$$

be the subset of M fixed by both g and h. Let $\mathcal{V}(g,h)$ be a normal bundle of $M^{(g,h)}$ in M.

We view (g,h) as a homomorphism

$$\Lambda_n \xrightarrow{(g,h)} G :$$

this makes Λ_n act trivially on $M^{(g,h)}$, and we let

$$e_t(\mathcal{V}(g,h)) \in E^*(B\Lambda_n) \otimes E^*(M^{(g,h)}) \cong E^*(E\Lambda_n \times_{\Lambda_n} M^{(g,h)})$$

be the Λ_n-equivariant euler class of $\mathcal{V}(g,h)$ in the orientation t.

Recall from (5.2) that $L = \operatorname{colim} L_n$ is the colimit of the rings L_n obtained from $E^*B\Lambda_n$ by inverting the euler classes of non-trivial characters of Λ.

Proposition 6.3. *Suppose that $(g,h) \neq (0,0)$.*

(1) *The euler class $e_t(\mathcal{V}(g,h))$ is a unit of*

$$L \otimes_{E^*} E^*(M^{(g,h)}).$$

(2) *The quantity*

$$1 \otimes \pi_t^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right)$$

lies in the subring $D \subset L$.

(3) *As elements of D we have*

$$\Xi_{g,h}(\pi_t^{M,G}(1)) = 1 \otimes \pi_t^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right).$$

Thus the orbifold genus of M associated to t is given by the formula

$$\Phi^t_{\mathrm{orb}}(M) = \Phi^t(M) + \sum_{\substack{gh=hg \\ (g,h)\neq(0,0)}} 1 \otimes \pi_t^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right). \tag{6.4}$$

Proof. Keeping in mind that $M^{(g,h)}$ is a compact manifold, the first assertion follows by the argument originally due to [AS68].

Now examine the diagram

$$\begin{tikzcd}
B\Lambda \times M^{(g,h)} \arrow[r, "i"] \arrow[d, "1\times\pi^{M^{(g,h)}}"'] & E\Lambda \times_{(g,h)} M \arrow[r, "j"] \arrow[d, "\pi^{M,\Lambda}"'] & EG \times_G M \arrow[d, "\pi^{M,G}"] \\
B\Lambda \arrow[r, equal] & B\Lambda \arrow[r, "B(g,h)"] & BG
\end{tikzcd}$$

The right square is a pull-back, so

$$B(g,h)^* \pi_t^{M,G} = \pi_t^{M,\Lambda} j^*.$$

It follows that

$$\Xi_{(g,h)} \pi_!^{M,G}(1) = \pi_t^{M,\Lambda}(1),$$

considered as an element of D.

The fixed-point formula asserts that

$$\pi_!^{M,\Lambda}(1) = 1 \otimes \pi_!^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right)$$

in L: but in fact we know that the left-hand side is an element of $D \subset L$. It follows that the right hand side is too, and

$$\Xi_{(g,h)} \pi_!^{M,G}(1) = 1 \otimes \pi_!^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right).$$

The rest is easy. $\square$

The formula (6.4) is the analogue for the t-genus of the orbifold elliptic genera of [DMVV97, BL02]. To see this, let A be the (abelian) subgroup of G generated by g and h, and suppose that $\mathcal{V}(g,h)$ decomposes as a sum

$$\mathcal{V}(g,h) \cong L_1 \oplus \cdots \oplus L_r$$

of complex line bundles, with A acting on L_i by the character χ_i. Let

$$e(\chi_i) \in E(BA)$$

be the euler class of the character χ_i, using the orientation t, and let

$$y_i \in E(M^{(g,h)})$$

be the (non-equivariant) euler class of the line bundle L_i. Then

$$e_t(\mathcal{V}(g,h)) = \prod_i y_i +_F e(\chi_i).$$

We can be even more explicit. Let

$$F(x,y) \in E[\![x,y]\!]$$

be the formal group law over E induced by the orientation t. If R is a complete local E-algebra, let us write $F(R)$ for the maximal ideal of R, considered as an abelian group using the power series F to perform addition.

Associating to a character $\lambda \in \Lambda_n^*$ its first chern class in E-theory using the orientation t defines a group homomorphism

$$\Lambda_n^* \to F(E(B\Lambda_n)),$$

which gives rise to a homomorphism

$$\Lambda_n^* \to F(D_n);$$

in fact, this is the "level structure" referred to in §5.

The dual of the epimorphism

$$\Lambda \to \Lambda_n \to A$$

is a monomorphism

$$A^* \to \Lambda_n^*$$

which composes with the level structure to give a homomorphism

$$\phi : A^* \to \Lambda_n^* \to F(D_n).$$

By construction,

$$\phi(\chi_i) = e(\chi_i).$$

If e_1, e_2 are a basis for Λ_n^*, then we can write

$$\chi_i = a_i e_1 + b_i e_2$$

in Λ_n^*, where $a_i, b_i \in \mathbb{Z}/n$. If $v_i = \ell(e_i)$ for $i = 1, 2$, then

$$\phi(\chi_i) = [a_i](v_1) +_F [b_i](v_2).$$

Our typical summand in the formula (6.4) for the orbifold genus becomes

$$1 \otimes \pi_!^{M^{(g,h)}} \left(\prod_i \frac{1}{y_i +_F [a_i](v_1) +_F [b_i](v_2)} \right). \tag{6.5}$$

It is customary to calculate expressions like (6.5) by using the topological Riemann-Roch formula to pass to ordinary cohomology. In fact this approach is not available in our situation. To do so, one introduces the exponential

$$\exp : \widehat{\mathbb{G}}_a \to F$$

of the group law F, and finds $x_i \in L \otimes E(M^{(g,h)})$ and $w_i \in \widehat{\mathbb{G}}_a(L)$ such that

$$\begin{aligned} y_i &= f(x_i) \\ v_1 &= f(w_1) \\ v_2 &= f(w_2). \end{aligned}$$

However, if $v_1 = f(w_1)$ then $0 = [n]_F(v_1) = f(nw_1)$, which implies that $nw_1 = 0$, and, as L is torsion free, we must have $w_1 = 0$!

Nevertheless, we shall proceed formally in order to compare our formula with those of [DMVV97, BL02]. We have

$$e_t(\mathcal{V}(g,h)) = \prod_i f(x_i + a_i w_1 + b_i w_2).$$

Let $u_j, j = 1, \ldots, r$ be the roots of the total Chern class of the tangent bundle of $M^{(g,h)}$:

$$c(M^{(g,h)}) = \prod_j (1 - u_j).$$

Then the Riemann-Roch formula gives

$$\pi_t^{M^{(g,h)}} \left(\frac{1}{e_t(\mathcal{V}(g,h))} \right) = \int_{M^{(g,h)}} \left(\prod_j \left(\frac{u_j}{f(u_j)} \right) \prod_i \frac{1}{f(x_i + a_i w_1 + b_i w_2)} \right). \tag{6.6}$$

In [BL02], Borisov and Libgober use the two-variable elliptic whose exponential is

$$f(x) = \frac{\theta(x,\tau)}{\theta(x - z, \tau)}, \tag{6.7}$$

where

$$\theta(x,\tau) = -iq^{\frac{1}{8}}(e^{\frac{x}{2}} - e^{-\frac{x}{2}}) \prod_{n\geq 1}(1-q^n)(1-q^n e^x)(1-q^n e^{-x}), \tag{6.8}$$

τ is a complex number with positive imaginary part, and $q = e^{2\pi i\tau}$. (We have adopted slightly different conventions regarding factors of 2π. The simplest way to compare is to say that we work with the elliptic curve $\mathbb{C}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z})$, while they work with the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$)

Their expression for the orbifold two-variable genus is

$$\mathrm{Ell}_{orb}(M,G) = \frac{1}{|G|}\sum_{gh=hg}\Phi_{g,h},$$

where, with our conventions,

$$\Phi_{g,h} = \int_{M^{(g,h)}}\left(\prod_j\left(\frac{u_j}{f(u_j)}\right)\prod_i\frac{1}{f(x_i + A_i(1/n) - B_i(\tau/n))}e^{zB_i/n}\right). \tag{6.9}$$

and the A_i and B_i are *integer* representatives of a_i and b_i.

This differs from the formal expression (6.6) by only the factors $e^{zb_i/n}$. These factors are familiar from the study of equivariant genera; for example they are analogous to the factors $\nu_s^{1/k}$ in (11.26) of [BT89], $S(c_1(\mathcal{V}^{1/n})_{\mathbb{T}})$ in (6.17) of [AB02], or $u^{\frac{k}{n}\hat{I}(\bar{m})}$ in (5.16) of [And03]. Their role is to make the expression (6.9) independent of the choice of representatives A_i and B_i. It is necessary to introduce these factors because f is not doubly periodic; instead we have

$$f(x + 2\pi i\ell + 2\pi ik\tau) = y^{-k}f(x), \tag{6.10}$$

where $y = e^z$, as one checks easily using (6.8). So the expression doesn't depend on the choice of A_i. If $B_i' = B_i + n\delta_i$, then

$$\begin{aligned}\prod_i f(x_i + A_i/n - B_i'\tau/n)e^{zB_i'/n} &= \prod_i f(x_i + A_i/n - B_i\tau/n)y^{-\sum\delta_i}e^{zB_i/n}y^{\sum\delta_i}\\ &= \prod_i f(x_i + A_i/n - B_i\tau/n)e^{zB_i/n}.\end{aligned}$$

This is not an issue in our expression (6.5), and so the factors $e^{zB_i/n}$ have no role in our genus.

7. Comparison with the analytic equivariant genus

In fact, we can use the expression (6.7) for the two-variable elliptic genus in terms of theta functions to construct a Thom class in Grojnowski's equivariant cohomology, following [Ros01, AB02, And03]. When $G = \mathbb{T}[n]$ is a cyclic group of order n acting on a compact manifold M, we can write down a formula for a $\mathbb{T}$-equivariant genus on $\mathbb{T}\times_G M$, which by Shapiro's lemma is a sensible notion of G-equivariant genus on M. When we do so, we obtain the formula of [DMVV97, BL02].

Let Λ be the lattice $(2\pi i\mathbb{Z}+2\pi i\tau\mathbb{Z})$, let C be the elliptic curve $\mathbb{C}/\Lambda$, and let $E_{\mathbb{T}}$ be Grojnowski's equivariant elliptic cohomology associated to C. For convenience we identify

$$\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$$

so that

$$G \cong \mathbb{Z}[\frac{1}{n}]/\mathbb{Z}.$$

We identify

$$\mathbb{T}\times\mathbb{T} \cong C$$

by the formula

$$(r+\mathbb{Z}, s+\mathbb{Z}) \mapsto 2\pi ir + 2\pi is\tau + \Lambda. \tag{7.1}$$

Let M be a G-manifold with an equivariant complex structure on its tangent bundle T. We define

$$E_G(M) \stackrel{\text{def}}{=} E_{\mathbb{T}}(\mathbb{T} \times_G M).$$

For $a \in C$,

$$(\mathbb{T} \times_G M)^a = 0$$

unless $a \in C[n]$, so $E_T(\mathbb{T} \times_G M)$ is a metropolitan sheaf (collection of skyscraper sheaves) supported at $C[n]$. The stalk at a point a of order dividing n is

$$H(EG \times_G X^a).$$

Let $\check{T}$ be the lattice of cocharacters in $Spin(2d)$. In [And03], the first author constructed orientations for theta functions

$$\Theta = \Theta(\,-\,,\tau) : \check{T} \otimes \mathbb{C} \to \mathbb{C}$$

satisfying

$$\Theta(x + 2\pi i\ell + 2\pi ik\tau, \tau) = \exp(-2\pi i I(k,x)) \exp(-\phi(k)) \Theta(x,\tau) \tag{7.2}$$

for $x \in \check{T} \otimes \mathbb{C}$ and $k, \ell \in \check{T}$, where

$$\begin{aligned} \phi :& \check{T} \to \mathbb{Z} \\ I :& \check{T} \times \check{T} \to \mathbb{Z} \end{aligned}$$

are respectively quadratic and bilinear functions related by

$$\phi(\ell + \ell') = \phi(\ell) + I(\ell, \ell') + \phi(\ell').$$

The building block of the orientation is a family of functions F which we now describe. For simplicity we have supposed that TM is a complex vector bundle, and so our structure group is $U(d)$ instead of $Spin(2d)$. We let T be the maximal torus of diagonal matrices; our choices so far identify the lattice $\check{T} = \hom(\mathbb{T}, T)$ of cocharacters with $\mathbb{Z}^d$ in the usual way.

Let $(g,h) \in G^2$; let a be the corresponding point of $C[n]$, and let $A \subseteq G$ be the subgroup generated by g and h. The action of A on $TM|_{M^A}$ is described by characters

$$m = (m_1, \ldots, m_d) \in \hom(A, T) \cong \check{T}/|A|\check{T} \cong (\mathbb{Z}/|A|)^d.$$

Choose integer lifts

$$\bar{m} = (\bar{m}_1, \ldots, \bar{m}_d) \in \check{T} \cong \mathbb{Z}^d.$$

Choose

$$\bar{a} = 2\pi i \frac{\ell}{n} + 2\pi i \frac{k}{n} \tau$$

so that

$$a = \bar{a} + \Lambda.$$

In terms of these choices, the stalk of the orientation at (g,h) is built from the function

$$F(x) = \exp(2\pi i \frac{k}{n} I(x, \bar{m})) \exp(2\pi i \frac{k}{n} \phi(\bar{m})\tau) \Theta(x + \bar{m} \otimes \bar{a})$$

The essential feature of F is that the functional equation (7.2) satisfied by Θ implies that F is Weyl invariant and independent of the choice of preimage $\bar{m}$, and its dependence on $\bar{a}$ is under control.

Now consider the exponential f (6.7) associated to the two-variable elliptic genus. Comparison of its functional equation (6.10) with the functional equation (7.2) for Θ suggests that we should build the orientation for the two-variable genus from the simpler function

$$F(x) = \exp(2\pi i \frac{k}{n} z \sum \bar{m}_j) \prod_j f(x_j + 2\pi i \bar{m}_j \ell/n + 2\pi i \tau \bar{m}_j k/n). \tag{7.3}$$

The argument at the end of §6 shows that indeed, this F is independent of the choice of lift $\bar{m}$. In fact, the simple transformation rule (6.10) for f implies that F is also independent of the choice of representative $\bar{a}$ for a.

Now use this F to write down a class $\mathrm{Ell}^{an}(M,G) \in E_G(M)$ following the instructions in [And03]. (In general one gets a section of the cohomology of the Thom space, but the Thom isomorphism in ordinary cohomology defined by f identifies this with $E_G(M)$). More precisely, the formula for the value in the stalk at $a \in C[n]$ is the one for γ_a before Lemma 8.12, taking V' to be trivial and θ' to be 1. That formula refers to an expression R which is defined in terms of F in Lemma 5.28. (The expression for R also includes a product of σ functions, which should be replaced with the corresponding product of f's).

Theorem 7.4. *With these substitutions, the value of* $\mathrm{Ell}^{an}(M,G)$ *in the stalk at* (g,h) *is the class in* $H(EG \times_G M^{(g,h)})$ *whose restriction to* $H(BA \times M^{(g,h)})$ *is the integrand in the summand* $\Phi_{g,-h}$ *of the orbifold two-variable elliptic genus (see* (6.9)*).* $\square$

With the remarks so far, the proof is straightforward. We omit the details, except note that if $(g,h) = (\ell/n + \mathbb{Z}, k/n + \mathbb{Z})$, then in (6.9), (A_i, B_i) can be taken to run over the set $(\ell \bar{m}_i, k\bar{m}_i)$. Thus the typical factor in (6.9) can easily be seen to identify with the typical factor in (7.3).

The genus associated to $\mathrm{Ell}^{an}(M,G)$ is the global section of $\mathcal{O}_{C[n]}$ whose value at (g,h) is

$$\int_{M^{(g,h)}} \mathrm{Ell}^{an}(M,G)_{g,h}.$$

Summing over $(g,h) \in G$, we get exactly $|G|$ times the orbifold two-variable elliptic genus of [DMVV97, BL02].

8. The cocycle

We now return to the orbifold sigma genus, and study the effect of varying the $BU\langle 6\rangle$ structure. The fibration of infinite loop spaces

$$K(\mathbb{Z},3) \to BU\langle 6\rangle \to BSU$$

gives a map of E_∞ ring spectra

$$\Sigma^\infty K(\mathbb{Z},3)_+ \xrightarrow{i} MU\langle 6\rangle.$$

If $\mathbf{C} = (E, C, t)$ is an elliptic spectrum, then the sigma orientation

$$\sigma(\mathbf{C}) : MU\langle 6\rangle \to E$$

gives rise to a map of ring spectra

$$w(\mathbf{C}) \stackrel{\text{def}}{=} \sigma(\mathbf{C}) i : \Sigma^\infty K(\mathbb{Z},3)_+ \to E.$$

In particular, $w(\mathbf{C})$ is a Thom class for the trivial bundle over $K(\mathbb{Z},3)$, and so it is a *unit* of $E^0 K(\mathbb{Z},3)$.

If $\alpha = (g,h) : \Lambda \to G$, then we define

$$\delta_n(\alpha) = \delta_n(u, \mathbf{C}, \alpha) \stackrel{\text{def}}{=} \Xi_{g,h}(u^* w(\mathbf{C})) \in D^\times. \tag{8.1}$$

Lemma 8.2. *If*

$$\alpha' = \Lambda_{n+1} \to \Lambda_n \xrightarrow{\alpha} G,$$

then

$$\delta_{n+1}(\alpha') = \delta_n(\alpha), \tag{8.3}$$

and so we have a well-defined unit $\delta(g,h) \in D^\times$*. It satisfies*

$$\begin{aligned}
\delta(g,h)^n &= 1 \\
\delta(h,g) &= \delta(g,h)^{-1} \\
\delta(g+g',h) &= \delta(g,h)\delta(g',h) \\
\delta(g,h+h') &= \delta_n(g,h)\delta(g',h)
\end{aligned}$$

for any $g, g', h,$ *and* h' *in* G *for which these equations make sense.*

Proof. The arguments for the various claims are similar to each other; as an illustration we show that δ is exponential in the first variable, and for simplicity we suppose that G is abelian. The proof in the general case will be given in §10.

Let C be the cyclic group of order n. The universal example of an abelian group with three elements (g_1, g_2, h) of order n is C^3. Let

$$C^2 \xrightarrow{(g_1,h)} C^3$$

$$C^2 \xrightarrow{(g_2,h)} C^3$$

$$C^2 \xrightarrow{(g_1+g_2,h)} C^3$$

be the maps which represent the selection of the indicated pairs elements formed from the triple (g_1, g_2, h). It suffices to show that for every homotopy class

$$u : BC^3 \to K(\mathbb{Z}, 3),$$

the outside rectangle of the diagram

$$\begin{array}{ccccccc} BC^2 & \xrightarrow{(g_1+g_2,h)} & BC^3 & \xrightarrow{u} & K(\mathbb{Z},3) & \xrightarrow{\sigma(\mathbf{C})} & E \\ {\scriptstyle\Delta}\downarrow & & & & \uparrow & & \uparrow \\ BC^2 \times BC^2 & \xrightarrow{(g_1,h)\times(g_2,h)} & BC^3 \times BC^3 & \xrightarrow{u\times u} & K(\mathbb{Z},3)\times K(\mathbb{Z},3) & \xrightarrow{\sigma(\mathbf{C})\times\sigma(\mathbf{C})} & E\times E \end{array} \tag{8.4}$$

commutes. The right-side rectangle commutes, because $\sigma(\mathbf{C})$ is a map of ring spectra. To show that the left-side rectangle commutes, consider the diagram

$$\begin{array}{ccccccc} BC^2 & \xrightarrow{(g_1+g_2,h)} & BC^3 & \xrightarrow{v} & K(C,2) & \xrightarrow{\beta} & K(\mathbb{Z},3) \\ {\scriptstyle\Delta}\downarrow & & & & \uparrow & & \uparrow \\ BC^2 \times BC^2 & \xrightarrow{(g_1,h)\times(g_2,h)} & BC^3 \times BC^3 & \xrightarrow{v\times v} & K(C,2)\times K(C,2) & \xrightarrow{\beta\times\beta} & K(\mathbb{Z},3)\times K(\mathbb{Z},3). \end{array} \tag{8.5}$$

Once again, the right square commutes, this time because the Bockstein is an additive group homomorphism.

An easy calculation shows that the Bockstein

$$\beta : H^2(BC^3; C) \to H^3(BC^3; \mathbb{Z})$$

is surjective, so there is a v such that $\beta v = u$. Moreover

$$H^2(BC^3; C) \cong C\{\mu_{12}, \mu_{13}, \mu_{23}\},$$

where

$$\mu_{ij} : BC^3 \cong K(C,1)^3 \to K(C,2)$$

is the map which represents the natural transformation

$$\mu_{ij} : H^1(X; C)^3 \to H^2(C; C)$$

given by

$$\mu_{ij}(x_1, x_2, x_3) = x_i \cup x_j.$$

If

$$v = a\mu_{12} + b\mu_{13} + c\mu_{23},$$

then the top row of the diagram represents the natural transformation

$$H^1(X; C)^2 \to H^3(X; \mathbb{Z})$$

given by

$$(x, y) \mapsto \beta(ax \cup x + bx \cup y + cx \cup y),$$

while the other (i.e. counterclockwise) composition represents the natural transformation

$$(x, y) \mapsto \beta(bx \cup y + cx \cup y).$$

These coincide since $\beta(x \cup x) = 0$.

In other words, the diagram (8.5) commutes for *any* v, and so the diagram (8.4) does too, as required. $\square$

8.1. **The Weil pairing.** As an example, let's consider the case that $G = \Lambda$, and

$$u : BG \to K(\mathbb{Z}/N, 2)$$

is the map representing the cup product: indeed u is a generator of $E^0 BG$. A homomorphism

$$\alpha : \Lambda \to G$$

gives rise to a homomorphism

$$G^* \to \Lambda^* \to C[N].$$

Thus we may view the homomorphism $\alpha = (g, h)$ as a pair of N-torsion points of $C[N]$. The argument of [AS01] then shows

Lemma 8.6. *$\delta(u)$ is the* Weil pairing *of the elliptic curve C.* $\square$

9. Discrete torsion

We are now ready to state our basic formula.

Theorem 9.1. *If $\alpha = (g, h) : \Lambda_n \to G$, then*

$$\Xi_\alpha \sigma(M, \ell + \iota u \pi, \mathbf{C})_G = \delta(u, \mathbf{C}, \alpha) \Xi_\alpha \sigma(M, \ell, \mathbf{C})_G,$$

and so abbreviating $\delta(g, h) = \delta(u, \mathbf{C}, \alpha)$, we have

$$\sigma_{\mathrm{orb}}(M, \ell + \iota u \pi, \mathbf{C})_G = \sum_{gh = hg} \delta(g, h) \Xi_{g,h} \sigma(M, \ell, \mathbf{C})_G \in \pi_{-d} E.$$

Proof. Since

$$\sigma(\mathbf{C}) : MU\langle 6 \rangle \to E$$

is a map of ring spectra, and since the multiplication on $MU\langle 6\rangle$ arises from the addition on $BU\langle 6\rangle$, we have

$$\begin{aligned} U(M, \ell + \iota u \pi, \mathbf{C})_G &= U(M, \ell, \mathbf{C})_G U(M, \iota u \pi, \mathbf{C})_G \\ &= U(M, \ell, \mathbf{C})_G \pi^* u^* w(\mathbf{C}). \end{aligned}$$

Recall that $E^0(V_G - T_G)$ is an $E^0(M_G)$-module, and so an $E^0(BG)$-module via $E^0(\pi)$. As such the Pontrjagin-Thom map

$$\tau(V)_G : E^0(V_G - T_G) \to E^0(BG)$$

is a homomorphism of $E^0(BG)$-modules. It follows that

$$\begin{aligned} \sigma(M, \ell + \iota u \pi, \mathbf{C})_G &= \tau(V)_G (U(M, \ell + \iota u \pi, \mathbf{C})_G) \\ &= \tau(V)_G (U(M, \ell, \mathbf{C})_G \pi^* u^* w(\mathbf{C})) \\ &= u^* w(\mathbf{C}) \sigma(M, \ell, \mathbf{C})_G \end{aligned}$$

in $E^*(BG)$. If

$$\alpha : \Lambda_n \to G,$$

then applying Ξ_α gives

$$\begin{aligned} \Xi_\alpha \sigma(M, \ell + \iota u \pi, \mathbf{C})_G &= \Xi_\alpha u^* w(\mathbf{C}) \Xi_\alpha \sigma(M, \ell, \mathbf{C})_G \\ &= \delta_n(u, \mathbf{C}, \alpha) \Xi_\alpha \sigma(M, \ell, \mathbf{C})_G. \end{aligned}$$

as required. $\square$

10. The non-abelian Case

In this section, we prove Lemma 8.2 in the case that G is non-abelian. We fix an n sufficiently large that $|G|$ divides n. We first construct an isomorphism $H^3(B\Lambda_n) \cong \mathbb{Z}/n$.

Consider the following commutative diagram, where the columns are universal coefficient exact sequences.

$$\begin{tikzcd}
\mathrm{Ext}^1(H_1(B\Lambda_n);\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Ext}^1(H_1(B\Lambda_n);\mathbb{Z}/n) \arrow[d] & \\
H^2(B\Lambda_n) \arrow[r] \arrow[d] & H^2(B\Lambda_n;\mathbb{Z}/n) \arrow[r] \arrow[d] & H^3(B\Lambda_n) \arrow[dl, dashed, "\cong"] \\
\mathrm{Hom}(H_2(B\Lambda_n);\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_2(B\Lambda_n);\mathbb{Z}/n). &
\end{tikzcd}$$

Here, the middle row is part of Bockstein long exact sequence. Note that this is a short exact sequence since multiplication by n kills group cohomology of Λ_n. Since $H_2(B\Lambda_n)$ is torsion, $\mathrm{Hom}(H_2(B\Lambda_n);\mathbb{Z}) = 0$. We therefore obtain the dotted arrow. Since $\mathrm{Ext}^1(H_1(B\Lambda_n);-)$ is right exact,the top map is a surjection. It follows that the dotted arrow is an isomorphism. Now it is not hard to check that $e_1 \otimes e_2 - e_2 \otimes e_1$ is a generator for $H_2(B\Lambda_n) \cong \mathbb{Z}/n$. Composing the dotted arrow above with evaluation on this generator yields an isomorphism

$$H^3(B\Lambda_n) \cong \mathbb{Z}/n.$$

Definition 10.1. If u is an element in $H^3(BG)$ and $\alpha = (g,h) : \Lambda_n \to G$ is any map, then we obtain a class $\alpha^* u \in H^3(B\Lambda_n)$. Let $\epsilon(g,h) = \epsilon^n_u(g,h)$ be the image of this class in $\mathbb{Z}/n$ under the isomorphism above.

Remark 10.2. It is easy to check that

$$\epsilon(g,h) = \tilde{u}(g,h) - \tilde{u}(h,g)$$

where $\tilde{u} : G \times G \to \mathbb{Z}/n$ is a 2-cocycle whose cohomology class maps to u under the Bockstein.

Lemma 10.3. *Whenever the expressions are defined, the following properties hold.*

$$\epsilon(g,h) = -\epsilon(h,g)$$

$$\epsilon(h,j) - \epsilon(gh,j) + \epsilon(g,hj) - \epsilon(g,h) = 0$$

$$\epsilon(gg',h) = \epsilon(g,h) + \epsilon(g',h)$$

$$\epsilon(g,hh') = \epsilon(g,h) + \epsilon(g,h')$$

Proof. The first and second properties follow easily from the remark. For the third property, it suffices to show that

$$\tilde{u}(gg',h) - \tilde{u}(h,gg') - \tilde{u}(g,h) + \tilde{u}(h,g) - \tilde{u}(g',h) + \tilde{u}(h,g')$$

is zero. Since $\tilde{u}$ is a cocycle, we may rewrite the expression using the following equations:

$$\begin{aligned}
\tilde{u}(gg',h) - \tilde{u}(g',h) &= \tilde{u}(g,g'h) - \tilde{u}(g,g'),\\
-\tilde{u}(h,gg') + \tilde{u}(h,g) &= \tilde{u}(g,g') - \tilde{u}(hg,g'),\\
-\tilde{u}(g,h) + \tilde{u}(h,g') &= \tilde{u}(gh,g') - \tilde{u}(g,hg').
\end{aligned}$$

Then, canceling terms, we get

$$\tilde{u}(g,g'h) + \tilde{u}(gh,g') - \tilde{u}(hg,g') - \tilde{u}(g,hg').$$

Since $gh = hg$ and $g'h = hg'$, this is zero as needed.

The last property follows similarly, or directly from the first and third. $\square$

Proof of Lemma 8.2. Let

$$F : B\Lambda_n \to K(\mathbb{Z}, 3)$$

represent the element $1 \in \mathbb{Z}/n \cong H^3(B\Lambda_n)$ under the isomorphism above. Let $x \in D$ be the image of

$$F^* w(\mathbf{C}) \in E^0 B\Lambda_n$$

under the map tautological map

$$E^0 B\Lambda_n \to D_n \to D.$$

It is easy to check from the definitions (8.1) and (10.1) of δ and ϵ that

$$\delta(g, h) = x^{\epsilon_{g,h}}.$$

The result now follows from Lemma 10.3. □

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